

REDUCED MODELS OF HIGHER ORDER SYSTEMS BASED ON KAUTZ EXPANSIONS

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Abstract. *In this contribution, the advantages of reduced order modeling (ROM) techniques based on oblique projectors are highlighted. Specific attention is focussed on obtaining models which have small residual errors over a predetermined bandwidth. As expansion functions, a bandlimited two-parameter Kautz basis is proposed. The proposed technique belongs to the class of general oblique projection techniques. Pertinent features of the method are the ability to preserve the structure of the original system and the fact that a more efficient reduced order modeling approach is obtained by focusing on the frequency band under scrutiny.*

1 INTRODUCTION

Most physical modeling algorithms boil down to some kind of discretization of the differential equations describing the problem at hand. The number of structures that can be analyzed in an analytical way, is limited. Hence, an appropriate numerical discretization technique has to be used yielding a set of linear equations as a result. These equations need to be solved for some or all of the variables. The unknowns are related to the discretized physical variables in the simulation domain. Reduced order modeling techniques make it possible to obtain much smaller descriptions than the original one for describing complex systems. This allows integration into an overall design and / or can be used for optimization purposes.

All reduced order modeling methods can be classified as projection methods.¹ A relatively new technique is the the bandlimited Laguerre² ROM method. This method has a number of advantages over the more traditional Krylov subspace methods.³⁻⁵ An important property of the bandlimited Laguerre technique, is that the number of inversions which need to be performed, is the number of quadrature points needed to calculate the projection subspace basis vectors instead of equal to the number of moments to be matched. With the Krylov subspace methods, it is not a priori known whether a memory

overflow could occur or not. With bandlimited Laguerre, a sufficient number of quadrature points is chosen a priori, determining the amount of memory that will be needed. An other advantage of the bandlimited approach is that it can easily be parallelized.

Oblique projection methods, such as the bandlimited Laguerre method, have interesting properties with respect to the preservation of structure^{6,7} of higher order systems.⁸ In this contribution, a generalization of the bandlimited Laguerre method is proposed. The transfer function is expanded in a different set of basis functions, called the Kautz basis. The Laguerre basis is a special case of the Kautz basis.

2 KAUTZ BASIS FUNCTIONS

Here, we consider a bandlimited Kautz basis, which is shown to be orthonormal over a narrowband frequency interval. By means of projecting the original transfer function onto this basis, we construct an oblique projector.^{1,9}

The two-parameter Kautz basis reads:

$$\begin{aligned}\phi_{2n}(s) &= \sqrt{2\tau} (s + \sqrt{\tau^2 + \sigma^2}) \frac{((s - \tau)^2 + \sigma^2)^n}{((s + \tau)^2 + \sigma^2)^{n+1}} \quad n = 0, 1, \dots \\ \phi_{2n+1}(s) &= \sqrt{2\tau} (s - \sqrt{\tau^2 + \sigma^2}) \frac{((s - \tau)^2 + \sigma^2)^n}{((s + \tau)^2 + \sigma^2)^{n+1}} \quad n = 0, 1, \dots\end{aligned}\quad (1)$$

These basis functions are a generalization of the scaled Laguerre functions,² which can be obtained by taking $\sigma = 0$. All two-parameter Kautz basis functions $\phi_n(s)$ exhibit the same frequency behavior in magnitude, i.e. we have for all n that

$$|\phi_n(i\omega)|^2 \stackrel{\text{def}}{=} \mathcal{M}(\omega) = \frac{2\tau(\omega^2 + \tau^2 + \sigma^2)}{(\tau^2 + \sigma^2 - \omega^2)^2 + 4\omega^2\tau^2} \quad (2)$$

The Kautz basis offers the advantage that its frequency dependent behavior resembles that of filter-like structures. Completely analogous to the bandlimited case,² we apply a coordinate transform which maps the Kautz basis $\phi_n(i\omega)$, which is orthogonal over $[0, +\infty]$ to a new basis $\psi_n(i\omega)$, orthogonal over a limited bandwidth $[\alpha, \beta]$:

$$\psi_n(s) = \rho(s) \phi_n(\eta(s)) \quad n = 0, 1, \dots \quad (3)$$

where

$$\eta(s) = \frac{\beta^2}{s} \frac{s^2 + \alpha^2}{s^2 + \beta^2} \quad (4)$$

and

$$\rho(s) = \beta \frac{s^2 + s\sqrt{\beta^2 + 2\alpha\beta - 3\alpha^2} + \alpha\beta}{s(s^2 + \beta^2)} \quad (5)$$

3 OBLIQUE PROJECTORS

In this section, we demonstrate how the properties of an oblique projector Q , which is a matrix function of U and V , which uniquely determine the matrices describing the reduced model, guarantee that a finite number of expansion coefficients of both the original and reduced transfer functions are approximately equal. The well known operator

$$Q_2 = u(v^T u)^{-1}v^T \quad (6)$$

projecting a vector in \mathcal{R}^2 onto $\text{span}(u)$, and parallel to $\text{span}(v^\perp)$, can be generalized¹⁰ to a general \mathcal{R}^n space. If the columns of a matrix U span a space S_U and the columns of V span the space S_V , the projection operator projecting onto S_U and parallel to S_V^\perp is determined by :

$$Q = U(V^T U)^{-1}V^T \quad (7)$$

If the matrix Q has nulspace \mathcal{N} and range \mathcal{R} , it can be proven¹⁰ that the spectral norm $\|Q\|_2$ satisfies :

$$\|Q\|_2 = 1/\sin \theta \quad (8)$$

where θ is the angle between S_u and S_w , defined by $\cos \theta = \max |v^T u|$, and where u and v are two unit vectors from the the range and the nullspace of Q respectively. Now consider the following operator with A being a complex matrix and V and U real :

$$Q_A = U(V^T A U)^{-1}V^T A \quad (9)$$

This operator is an idempotent, and the range and the nullspace of this operator form complementary subspaces in \mathcal{C}^n . We suppose that $V^T A U$ is nonsingular. If we define $X^H = V^T A$, we can conclude that Q_A is the operator projecting onto $\text{colspan}(U)$ and parallel to $\text{colspan}(X)^\perp$. Now let us define the matrix transfer functions of the form

$$F(s) = L^T P(s)^{-1} B \quad (10)$$

where $P(s)$ is a matrix polynomial, i.e.:

$$P(s) = \sum_0^m A_k s^k \quad (11)$$

In order to obtain the matrices V and W we expand $P(s)^{-1}B$ in an orthonormal basis $\psi_k(s)$. Accordingly, we obtain :

$$P(s)^{-1}B = \sum_{k=0}^{r-1} k_k \psi_k(s) + R_r(s) \quad (12)$$

$$P(s)^{-T}L = \sum_{k=0}^{r-1} l_k \psi_k(s) + R_l(s) \quad (13)$$

where $R_l(s)$ and $R_r(s)$ are error terms. The k_k and the l_k can be calculated by fourier integrals and the coefficient matching is guaranteed by the properties of the associated idempotent $Q_{P(s)}$, which is shown below. Next, the $N \times q$ matrices K_r and K_l are defined as

$$\begin{aligned} K_r &= [k_0, k_1, \dots, k_{r-1}] \\ K_l &= [l_1, l_2, \dots, l_{r-1}] \end{aligned} \tag{14}$$

The columns of these two matrices K_r and K_l span the *right subspace* and the *left subspace* respectively. It can be proven¹¹ that if K_l, K_r are such that $\det(K_l^T K_r) \neq 0$, there exists an idempotent Q such that

$$\begin{aligned} QK_r &= K_r \\ Q^T K_l &= K_l \\ Q &= VW^T \\ W^T V &= I_q \end{aligned} \tag{15}$$

If $K_l^T K_r$ is nonsingular, then $Q_I = VW^T$, with $W^T V = I$ and I the identity matrix, such that $Q_I K_r = K_r$. Hence :

$$Q_I k_k = k_k \quad k = 0, \dots, r-1 \tag{16}$$

Note that $Q_{P(s)} Q_I = Q_I$. Multiplying the first equation of (12) with $Q_{P(s)}$ yields:

$$V(W^T P(s)V)^{-1} W^T P(s) P^{-1}(s) B = V(W^T P(s)V)^{-1} W^T B = \sum_{k=0}^{r-1} k_k \psi_k(s) + Q_{P(s)} R_r(s) \tag{17}$$

Next, we left multiply (17) with L^T , and consequently obtain the following transfer functions :

$$F(s) = \sum_{k=0}^{r-1} L^T k_k \psi_k(s) + L^T R_r(s) \tag{18}$$

$$F_R(s) = \sum_{k=0}^{r-1} L^T k_k \psi_k(s) + L^T Q_P(s) R_r(s) \tag{19}$$

Using (8), we observe that, when the norm of R_r is small enough, and when the angle θ is close to $\frac{\pi}{2}$, the first q expansion coefficients of the transfer functions $F(s)$ and $F_R(s)$ are approximately identical.

4 EXAMPLES

Two examples will be discussed in this section. The first example investigates how the frequency dependent convergence of the Kautz method behaves against the Multipoint Padé method. The second example shows a second order mechanical system, which is reduced by choosing basis functions which mimic the resonant behavior of the system.

4.1 CONVERGENCE STUDY

As a first comparison of the proposed method with other reduction techniques, we compared a reduction of an electromagnetic transmission line example, reduced with both the Kautz and the Multipoint Padé¹ algorithms. The original system (with the length of the vector of internal variables being 142905) was obtained by spatially discretizing Maxwell's equations according to a central difference scheme. For both reduction methods, the same frequency points were chosen in order to have a sound comparison. A number of 30 equidistant quadrature points is chosen in order to calculate the k_k and the l_k .

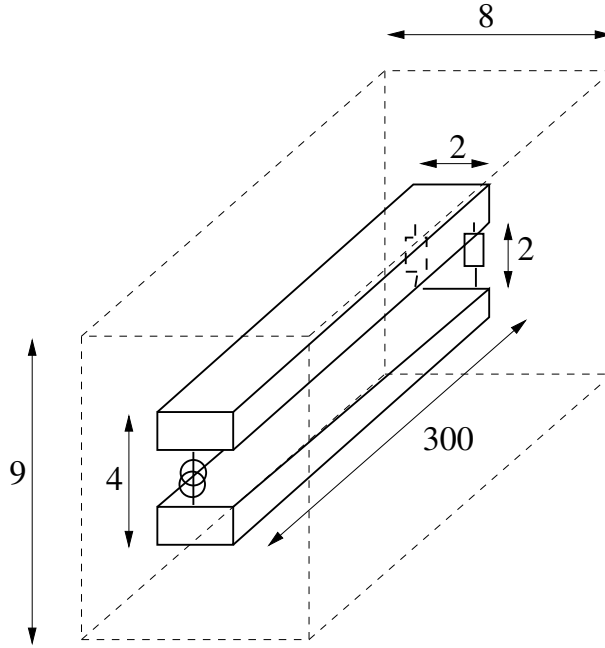


Figure 1: Transmission line example, analyzed with a finite difference analysis. The dimensions are shown in units of $0.1\mu m$. The structure is terminated with two parallel resistors of 20Ω .

The input impedance of the structure Z_i is modeled between $\alpha = 4\text{ GHz}$ and $\beta = 6\text{ GHz}$. The other parameters τ and σ are chosen to obtain the bandlimited Laguerre basis² ($\sigma = 0, \tau = \tau_{BL} = \sqrt{\beta(\alpha + \beta)/2}$). In Fig. 2 the relative L_1 error norms $\mathcal{E}_R(Z_i(q = q_i), Z_i(q = 60), f)$ of the real parts of the transfer functions with respect to their 'converged' values at $q = 60$, are presented as a function of frequency.

It is apparent that the Kautz algorithm is able to provide smaller maximal errors for equal reduction orders, and that the error is in general much less dependent on the frequency.

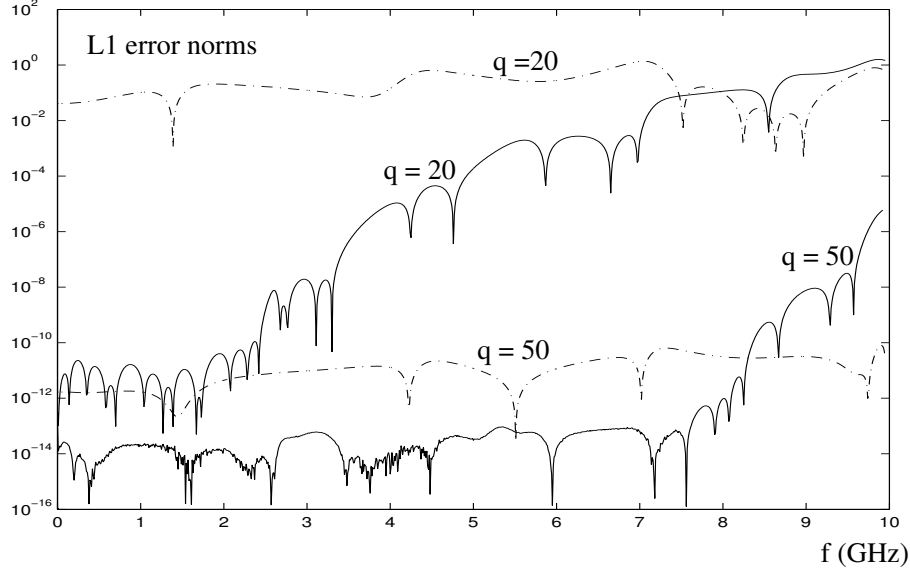


Figure 2: Relative L_1 error norms of the real part of the input impedance obtained with bandlimited Kautz ROM ($q = 20$ and $q = 50$, dash-dot line, $\tau = \tau_{BL}, \sigma = 0$) with Multipoint Padé (full line). Errors are relative to the values at $q = 60$.

4.2 STRUCTURE PRESERVATION OF A MECHANICAL SYSTEM

As an example of a higher order system, we will discuss a model of a large building, in our case a large city hall¹⁵ in Los Angeles. The model is derived by applying the finite element method to the equations governing the dynamics of the motion of the building. The system describes the components of the cartesian position vector \mathbf{r} , collected in a vector x , as a function of an external force vector f . The equations describing the building's dynamics can be written as:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t) \quad (20)$$

Here M is the mass matrix, C is the damping matrix, K is the stiffness matrix and $f(t)$ is the external force vector. Since the K and M matrices are related to the kinetic and strain energy respectively, they are symmetric and positive definite.

The bandwidth under scrutiny is $[\alpha, \beta] = [0.01\text{Hz}, 100\text{Hz}]$. The other simulation parameters are $\sigma = \tau = 10$ and the number of quadrature points, from which k_k and l_k are calculated, is 200. The original M , C and K matrices are of size 26294×26394 . Since orthogonal projections guarantee the conservation of positive definiteness, the pertinent reduced matrices are also positive definite. The C matrix accounts for the damping properties, and it is modeled as a linear function of M and K : $C = \mu M + \kappa K$ with typical constants $\mu = 0.675$ and $\kappa = 0.00315$. The real part of the response of the building sys-

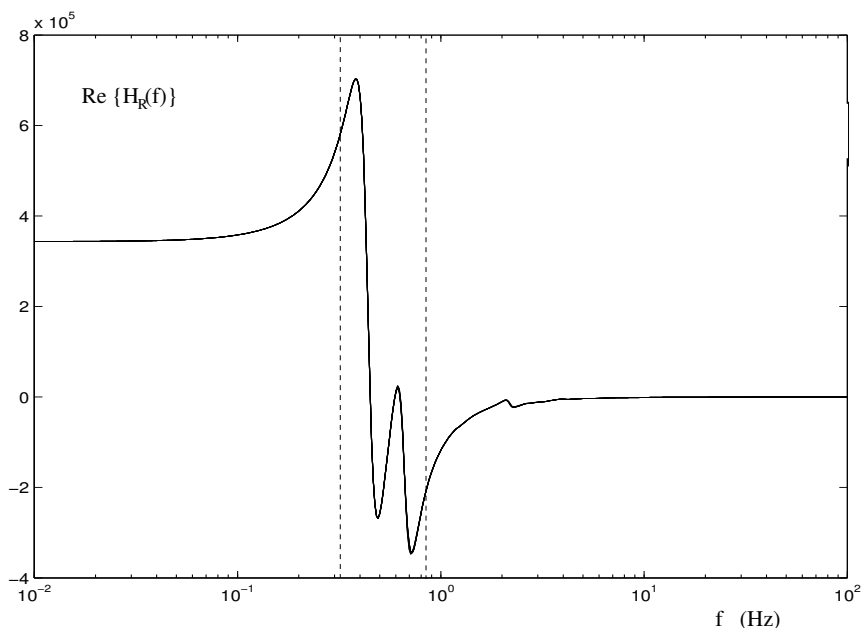


Figure 3: Real part of the transfer function of the building model, for $80 \leq q \leq 87$, together with the transfer function of the unreduced system.

tem is shown in Fig. 3, where the reduced transfer functions are shown for $80 \leq q \leq 87$, together with the transfer function of the unreduced system.

5 CONCLUSIONS

An important advantage of the oblique projection method is that the two projection matrices involved contain only half the number of columns as compared with the number of columns in the simpler orthogonal projection methods with one projection matrix. The technique is able to reduce a polynomial n -th order system into a polynomial n -th order system of lower degree, in a predetermined bandwidth.

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